Contents

1 Definitions

Let us define Lie Algebra $\mathfrak{sl}(2,\mathbb{C})$. Formally, it is a 3-dimensional complex Lie Algebra of all 2×2 complex matrices with trace zero. Let us recall the commutator for Lie algebras:

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

This commutator is skew-symmetric:

$$[x,y] = -[y,x]$$

Let us construct the standard basis:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As an aside, I note that $\mathfrak{su}(2) \otimes \mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C})$, where we 'complexify' $\mathfrak{su}(2)$ by extending the scalars from $\mathbb{R}$ to $\mathbb{C}$, although they tend to use different bases. $\mathfrak{su}(2)$ is a Real Lie algebra of unitary 2x2 traceless matrices with complex entries, but the linear combinations allowed within elements of $\mathfrak{su}(2)$ must use real scalars. Here, we mark how the commutator operates on the basis of $\mathfrak{sl}(2,\mathbb{C})$.

$$[E,F] = H, \ [H,E] = 2E, \ [H,F] = -2F$$

Recall that a representation of $\mathfrak{sl}(2,\mathbb{C})$ will be a vector space V with homomorphism $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$. $\mathfrak{gl}(V)$ is the Lie Algebra of all linear transformations on vector space V, with Lie bracket $[A,B] = AB - BA$.

Take some vector $v \in V$, and some $x \in \mathfrak{sl}(2,\mathbb{C})$. We will write xv instead of $\rho(x)v$.

2 Representations of $\mathfrak{sl}(2, \mathbb{C})$

Theorem. Any representation of $\mathfrak{sl}(2, \mathbb{C})$ is completely reducible.

Proof. We will sketch this proof. Due to a lemma that I won't prove, we know that categories of complex representations of real Lie algebras and their 'complexifications' are equivalent. Since $\mathfrak{sl}(2, \mathbb{C})$ is the complexification of $\mathfrak{su}(2)$, their representations are the same. We know from other results of representation theory that representations of $\mathfrak{su}(2)$ are the same as representations of the compact Lie Group $SU(2)$. One final theorem possible with Haar measure on compact Lie groups is that any finite-dimensional representation of a compact Lie group is unitary and thus completely reducible. We conclude that $\mathfrak{sl}(2, \mathbb{C})$ is completely reducible. □

For this lecture, we will classify irreducible representations.

Definition. Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$. A vector $v \in V$ is a vector of weight $\lambda \in \mathbb{C}$ if it is an eigenvector for H with eigenvalue λ.

$$Hv = \lambda v$$

Denote the subspace of vectors of weight λ by $V[\lambda] \subset V$. **This subspace is importantly not a subrepresentation of V** as we will later see. We begin with two lemmas.

Lemma.

$$EV[\lambda] \subset V[\lambda + 2]$$
$$FV[\lambda] \subset V[\lambda - 2]$$

Proof. Take some $v \in V[\lambda]$. We want to show that $Ev \in V[\lambda + 2]$.

Note that $[H, E] = HE - EH$, so $[H, E]v = HEv - EHv$, or

$$H(Ev) = [H, E]v + EHv = 2Ev + \lambda Ev = (\lambda + 2)Ev$$

$$\therefore Ev \in V[\lambda + 2]$$

Now, show that $Fv \in V[\lambda - 2]$. Using the same logic as before,

$$HFv = [H, F]v - FHv = -2Fv - \lambda Fv = (\lambda - 2)Fv$$

$$\therefore Fv \in V[\lambda - 2]$$

□

Theorem. Every finite-dimensional representation V of $\mathfrak{sl}(2, \mathbb{C})$ can be written in the form

$$V = \bigoplus_{\lambda} V[\lambda]$$

We call this the **weight decomposition** of V.

Proof. First, we assume that V is irreducible since every finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is completely reducible. Define subspace $V' = \sum_{\lambda} V[\lambda] \subset V$, spanned by eigenvectors of H with weight λ.

Eigenvectors with different eigenvalues are linearly independent, so we can take the direct sum to be equivalent to V':

$$V' = \bigoplus_{\lambda} V[\lambda]$$

We know that V' is invariant under the action of H because that's where we get our eigenvectors v, and our lemma demonstrated that it is also invariant under the action of E and F. V' is thus a subrepresentation of V.

We know that V' is nonzero due to the fact that H has at least one eigenvector. Since we assumed that V is irreducible, we conclude that $V' = V$. This means that every vector in our representation is an eigenvector for H, and we know how our basis vectors E and F act on v. □

3 Classification of Irreducible Finite-Dimensional Representations

Now, assume that V is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, and take a 'maximal' weight λ of V:

$$\text{Re } \lambda \geq \text{Re } \lambda'$$

for every weight λ of V. We call this the **highest weight** of V. The vectors $v \in V[\lambda]$ will be the highest weight vectors.

Lemma. Let $v \in V[\lambda]$ be a highest weight vector in V.

1. $Ev = 0$
2. If $v^k = \frac{F^k}{k!}v,\ k \geq 0$, then
   (a) $Hv^k = (\lambda - 2k)v^k$
   (b) $Fv^k = (k+1)v^{k+1}$
   (c) $Ev^k = (\lambda - k + 1)v^{k-1},\ k > 0$

Proof. Since $Ev \in V[\lambda + 2]$, but λ is a highest weight, we know that $V[\lambda+2] = 0 \implies Ev = 0$.

I will only prove the last formula by induction using the previous results. For $k = 1$, we note that

$$Ev^1 = E(Fv) = [E,F]v + FEv = hv + 0 = \lambda v \ \checkmark$$

Assume this holds for k, show that it holds for $k+1$:

$$Ev^{k+1} = \frac{1}{k+1}EFv^k = \frac{1}{k+1}(Hv^k + FEv^k)$$

$$= \frac{1}{k+1}\left((\lambda - 2k)v^k + (\lambda - k + 1)Fv^{k-1}\right)$$

$$= \frac{1}{k+1}(\lambda - 2k + (\lambda - k + 1)k)v^k = (\lambda - k)v^k = (\lambda - (k+1) + 1)v^{k+1-1}$$

□

Lemma. Let $\lambda \in \mathbb{C}$. Define M_λ to be the infinite-dimensional vector space with basis $v^0, v^1, \ldots$

1. With $Ev^0 = 0$, we can define on M_λ the structure of an infinite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$.
2. If V is an irreducible finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ which contains a non-zero highest weight vector of highest weight λ, then $V = M_\lambda/W$ for some subrepresentation W.

Proof. I skip the proof for this lecture. □

Theorem. For any $n \geq 0$, let V_n be the finite-dimensional vector space with basis $v^0, v^1, \ldots, v^n$. Define the action of $\mathfrak{sl}(2,\mathbb{C})$ as follows:

$$Hv^k = (n - 2k)v^k$$

$$Fv^k = (k+1)v^{k+1},\ k < n;\ Fv^n = 0$$

$$Ev^k = (n + 1 - k)v^{k-1},\ k > 0;\ ev^0 = 0$$

Then, V_n is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$. This is the **irreducible representation with highest weight** n.

For $n \geq m$, representations V_n and V_m are non-isomorphic.

Every finite-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to one of the representations V_n.

Proof. Consider M_λ from above. If $\lambda = n$ is a non-negative integer, consider the subspace $M' \subset M_n$ spanned by $v^{n+1}, v^{n+2}, \ldots$. Then this is a subrepresentation stable under H and F (by the definition of how we set up v as the highest weight vector).

The only nontrivial relation to check is that $Ev^{n+1} \in M'$:

$$Ev^{n+1} = (n + 1 - (n + 1))v^n = 0$$

We can create a quotient space M_n/M' which is a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$, with basis $v^0, \ldots, v^n$ with the action of $\mathfrak{sl}(2, \mathbb{C})$ given above.

Any subrepresentation must be spanned by some subset of $v^0, v^1, \ldots, v^n$, but every subrepresentation then generates the entire representation V_n under the action $\mathfrak{sl}(2, \mathbb{C})$. We conclude that V_n is an irreducible finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$, where $\dim V_n = n + 1$ and hence not isomorphic to V_m where $n \neq m$.

Now, we must show that every irreducible representation is of the form V_n. Let V be an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ and let $v \in V[\lambda]$ be a highest weight vector. We then know that V is a quotient of M_λ from our previous lemma. Put differently, the vectors $v^k = \frac{f^k}{k!}v$ span it.

The v^k's all have different weights, and so if they are non-zero they must be linearly independent. We know that V is finite-dimensional on the otherhand and so only finitely many of the v^i are non-zero in M_λ. We take n be the maximual such that $v^n \neq 0$, which means that $v^{n+1} = 0$. In this case, $v^0, \ldots, v^n$ are all non-zero and have different weights which means that they are linearly independent and form a basis in V.

$v^{n+1} = 0 \implies Ev^{n+1} = 0$. We also have $Ev^{n+1} = (\lambda - n)v^n$, which implies that $\lambda = n$ is a non-negative integer and is of the form outlined above. □

Some pretty immediate and beautiful corollaries are as follows:

Corollary. Let V be a finite-dimensional complex representation $\mathfrak{sl}(2, \mathbb{C})$.

1. V admits a weight decomposition with integer weights, and highest weight n:

$$V = \bigoplus_{n \in \mathbb{Z}} V[n]$$

2. $\dim V[n] = \dim V[-n]$, and for $n \geq 0$, the maps

$$E^n : V[n] \to V[-n]$$

$$F^n : V[-n] \to V[n]$$

are isomorphic.